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COUPLED VIBRATION ANALYSIS OF PIEZOELECTRIC CERAMIC DISK RESONATORS

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The coupled vibration of piezoelectric ceramic disk resonators was analyzed using an analytical method whilst considering the piezoelectric effect. By introducing a mechanical coupling coefficient, the coupled vibration of the disk resonators was divided into two equivalent vibrations, which are the equivalent longitudinal and radial vibrations. By basing them on piezoelectric and motion equations, the expressions for the admittance of the piezoelectric ceramic disk resonator in coupled longitudinal and radial vibrations were derived, and the resonance frequency equations of the resonator in coupled vibration were obtained. Some important electro–mechanical parameters, such as the electro–mechanical coupling coefficient, were analyzed and their expressions are given when the coupling between the longitudinal and radial vibrations was considered. Some limiting vibrational modes, such as the one-dimensional thickness extensional vibration and the plane radial vibration of thin disk resonators, can be obtained directly from the theory developed in this paper. Experiments were carried out to demonstrate the proposed theory for the coupled vibration of disk resonators. The resonance frequencies of the resonators were measured. It is shown that the measured resonance frequencies are in good agreement with the computed results from the theory and two kinds of resonance frequencies can be obtained, which correspond to the coupled longitudinal and radial vibrations of the disk resonators.

1. INTRODUCTION

Disk resonators of piezoelectric ceramics are widely used in ultrasonic transducers, ceramic filters, vibration sensors and other applications. In traditional vibration analysis theory, the disk resonators are usually considered as one-dimensional and the coupling between the longitudinal and radial vibrations is neglected. To satisfy the requirement of one-dimensional theory, the thickness of the disk resonator must be much larger or smaller than its radius, and these two kinds of geometrical shapes of the resonator are the limiting cases of the slender rod and the thin disk. The analysis theory for the longitudinal vibration of the piezoelectric ceramic slender rod and the radial vibration of the thin circular disk has been well established and used in the design of resonators and measurement of piezoelectric

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ceramic material parameters [1–5]. However, in practical applications, the disk resonator has the finite dimensions of the thickness and radius. In these cases, the vibration of the finite dimension disk resonator is complex; the coupling between vibrational modes in the resonator must be taken into account, and one-dimensional theory is no longer applicable to these cases, especially for the disk whose thickness and radius have the same dimensional magnitude.

For the coupled vibration of a piezoelectric ceramic circular disk resonator of finite dimension, since one-dimension analysis theory produces a large error in the analysis of the resonance frequency and the vibrational mode, new analysis theory must be developed. With the development of electronic computers and computation technology, numerical methods have been used to analyze the complex coupled vibration of the piezoelectric ceramic circular disk resonators of finite dimension [6–12]. Since numerical methods obtain not only the resonance frequency and the corresponding vibration displacement distribution, but also the stresses and strains in the resonators, they have been used widely in engineering and technology. However, as a larger amount of data must be processed, numerical methods are sometimes cumbersome and time-consuming in the analysis and calculation of resonators, especially in the analysis of circular disk resonators used for ultrasonic transducers whose resonance frequency and vibrational mode are important in its design and performance improvement. In this paper, an analytical method has been developed for the coupled vibration of disk resonators [13–14]. Having a basis of the piezoelectric and motion equations of circular disk resonators, and ignoring the shearing stress and strain, the complex coupled vibration of finite dimension disk resonator was studied. Expressions for some important electro–mechanical parameters, such as the electric admittance and the electro–mechanical coupling coefficient, have been given when the coupling between different vibrational modes was considered, and the resonance frequency equations were obtained. Some limiting vibrational modes, such as the longitudinal vibration in a piezoelectric ceramic slender rod, the thickness extensional vibration and the plane radial vibration in a thin circular disk, have been analyzed and can be obtained from the theory of this paper. By comparison with numerical methods, the analytical method is simple and concise in the calculation of the resonance frequency and the analysis of vibrational modes. In the following, the method and the analysis for the piezoelectric ceramic disk resonators of finite dimension will be described.

2. ANALYSIS OF PIEZOELECTRIC CERAMIC DISK RESONATORS IN COUPLED VIBRATION

The coupled vibration of a piezoelectric ceramic disk resonator of thickness l and radius a was studied in this paper. The theoretical analysis model and the coordinate systems were shown in Figure 1. In cylindrical co-ordinates, the axes z and r are in the direction of the thickness and radius. The polarization and the external exciting electric field are along the z-axis. Since the polarization direction is parallel to that of the exciting electric field, the vibration of the resonator may be regarded as symmetrical vibration, and the shearing stress and strain can be ignored. In cylindrical co-ordinates, the piezoelectric and motion equations are

$$S_r = s_{11}^E T_r + s_{12}^E T_\theta + s_{13}^E T_z + d_{31} E_3, \qquad S_\theta = s_{12}^E T_r + s_{11}^E T_\theta + s_{13}^E T_z + d_{31} E_3, \tag{1, 2}$$

$$S_z = s_{13}^E (T_r + T_\theta) + s_{33}^E T_z + d_{33} E_3, \qquad D_3 = d_{31}(T_r + T_\theta) + d_{33} T_z + \epsilon_{33}^T E_3 \tag{3, 4}$$

$$\rho\, \partial^2 U_r / \partial t^2 = \partial T_r / \partial r + (T_r - T_\theta)/r, \qquad \rho\, \partial^2 U_z / \partial t^2 = \partial T_z / \partial z. \tag{5, 6}$$

Here, S_r, S_θ, S_z and T_r, T_θ, T_z are the strains and stresses in the radial, tangential and longitudinal directions, $s_{ij}(i, j = 1, 2, 3)$ are the elastic compliance coefficients in a constant electric field, d_{31} and d_{33} are the piezoelectric constants, E_3 and D_3 are the electric field intensity and the electric flux density, ϵ_{33}^T is the dielectric constant, and U_r, U_z are the radial and longitudinal displacements. Since the edge effect of the electric field and the shearing strain and stress are ignored, T_{rz}, $T_{r\theta}$, $T_{\theta z}$, S_{rz}, $S_{r\theta}$, $S_{\theta z}$, E_1, E_2, D_1, D_2 can be ignored, and the expressions of $\partial D_3/\partial z = 0$ and $\partial E_3/\partial r = 0$ are applicable to this case. On the other hand, as the extensional vibration in the resonator is predominant, the shear and torsion can be ignored, and therefore, the tangential displacement U_θ is also ignored. The relation between the strain and displacement is

$$S_r = \partial U_r/\partial r, \qquad S_\theta = U_r/r, \qquad S_z = \partial U_z/\partial z. \tag{7}$$

2.1. THE EQUIVALENT RADIAL VIBRATION OF THE PIEZOELECTRIC CERAMIC DISK RESONATOR IN COUPLED VIBRATION

From equations (1) and (2), the following can be obtained:

$$S_r - S_\theta = (s_{11}^E - s_{12}^E)(T_r - T_\theta), \quad S_\theta + S_r = (s_{11}^E + s_{12}^E)(T_r + T_\theta) + 2 s_{13}^E T_z + 2 d_{31} E_3. \tag{8, 9}$$

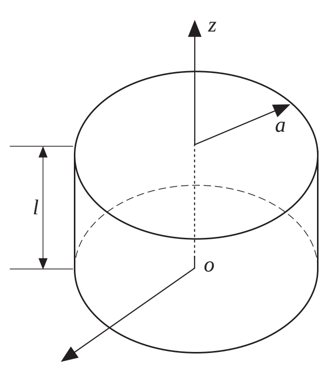

Figure 1. The theoretical analysis model and the coordinate systems

Let $n = T_z/(T_r + T_\theta)$. n is defined as the mechanical coupling coefficient between different vibrational modes. Equation (9) can be rewritten as

$$S_r + S_\theta = (s_{11}^E + s_{12}^E + 2s_{13}^E n)(T_r + T_\theta) + 2d_{31}E_3. \tag{10}$$

From equations (8) and (10), one has

$$T_r + T_\theta = (S_r + S_\theta - 2d_{31}E_3)/(s_{11}^E + s_{12}^E + 2s_{13}^E n), \tag{11}$$

$$T_r = \tfrac{1}{2}((S_r - S_\theta)/(s_{11}^E - s_{12}^E) + (S_r + S_\theta - 2d_{31}E_3)/(s_{11}^E + s_{12}^E + 2s_{13}^E n)). \tag{12}$$

Substituting the expressions of T_r and $(T_r - T_\theta)$ into equation (5) yields

$$\rho\, \partial^2 U_r/\partial t^2 = E_r(\partial^2 U_r/\partial r^2 + (1/r)\, \partial U_r/\partial r - U_r/r^2). \tag{13}$$

Here

$$\begin{aligned} E_r &= \tfrac{1}{2}(1/(s_{11}^E - s_{12}^E) + 1/(s_{11}^E + s_{12}^E + 2s_{13}^E n)) \\ &= (s_{11}^E + s_{13}^E n)/(s_{11}^E - s_{12}^E)(s_{11}^E + s_{12}^E + 2s_{13}^E n) \end{aligned}$$

is called the equivalent elastic constant in radial vibration. For harmonic vibration, $U_r = U_{ra}\exp(\mathrm{j}\omega t)$. The solution to equation (13) is

$$U_{ra} = A_r \mathrm{J}_1(k_r r) + B_r \mathrm{Y}_1(k_r r). \tag{14}$$

Here, $k_r = \omega/V_r$, $V_r = (E_r/\rho)^{1/2}$ and k_r and V_r are the equivalent wave number and sound speed in radial vibration. $\mathrm{J}_1(k_r r)$ and $\mathrm{Y}_1(k_r r)$ are Bessel functions. When $r = 0$, $\mathrm{Y}_1(k_r r)$ is divergent. Therefore, the displacement functions for the disk resonator is

$$U_{ra} = A_r \mathrm{J}_1(k_r r). \tag{15}$$

For free boundary condition, when $r = a$, $T_r = 0$, the constant A_r can be obtained as

$$A_r = \frac{(s_{11}^E - s_{12}^E)d_{31}E_3}{k_r \mathrm{J}_0(k_r a)(s_{11}^E + s_{13}^E n) - \mathrm{J}_1(k_r a)(s_{11}^E + s_{12}^E + 2s_{13}^E n)/a}\,. \tag{16}$$

On the other hand, from equations (4) and (10) the electric flux density D_3 can be obtained:

$$D_3 = \epsilon_{33}^T E_3 + [(d_{31} + nd_{33})/(s_{11}^E + s_{12}^E + 2s_{13}^E n)][A_r k_r \mathrm{J}_0(k_r r) - 2d_{31}E_3]. \tag{17}$$

Let the current into the resonator and the voltage across it be I_{31} and V_{31}. One has

$$V_{31} = \int_0^l E_3\, \mathrm{d}z = E_3 l, \qquad I_{31} = \mathrm{j}2\pi\omega \int_0^a D_3 r\, \mathrm{d}r. \tag{18, 19}$$

Substituting equation (17) and the expression of A_r into equation (19) yields

$$I_{31} = \mathrm{j}\omega\pi a^2 \epsilon_{33}^T E_3 \times \left[1 - (K_p')^2 + (K_p')^2 \frac{(s_{11}^E - s_{12}^E)\mathrm{J}_1(k_r a)}{k_r a \mathrm{J}_0(k_r a)(s_{11}^E + s_{13}^E n) - \mathrm{J}_1(\mathrm{k_r a})(s_{11}^E + s_{12}^E + 2s_{13}^E n)}\right]. \tag{20}$$

Here

$$(K_p')^2 = 2d_{31}(d_{31} + nd_{33})/[(s_{11}^E + s_{12}^E + 2s_{13}^E n)\epsilon_{33}^T].$$

K_p' is defined as the plane electro-mechanical coupling coefficient of the piezoelectric ceramic disk resonator in coupled vibration and can be rewritten as

$$(K_p')^2 = K_p^2(1 - n/\lambda_{31})/[1 - 2v_{13}n/(1 - v_{12})]. \tag{21}$$

where $K_p^2 = 2d_{31}^2/[\epsilon_{33}^T(s_{11}^E + s_{12}^E)]$, K_p is the plane electro-mechanical coupling coefficient of a thin disk resonator in ideal radial vibration, $\lambda_{31} = -d_{31}/d_{33}$, $v_{12} = -s_{12}^E/s_{11}^E$, $v_{13} = -s_{13}^E/s_{11}^E$. Using these expressions, the equivalent elastic constant of the disk resonator in radial vibration can be obtained:

$$E_r = (1 - v_{13}n)/[s_{11}^E(1 + v_{12})(1 - v_{12} - 2v_{13}n)]. \tag{22}$$

When $n = 0$, equation (21) and (22) can be expressed as

$$K_p' = K_p, \qquad E_r = 1/s_{11}^E(1 - v_{12}^2). \tag{23, 24}$$

It is obvious that in this case, the coupled vibration of the disk resonator is reduced to the plane radial vibration of a thin disk resonator. The equivalent radial admittance of the disk resonator in coupled vibration is

$$Y_{31} = I_{31}/V_{31} = \mathrm{j}\omega\pi a^2\epsilon_{33}^T/l \times \left[1 - (K_p')^2 + (K_p')^2 \frac{(1 + v_{12})\mathrm{J}_1(k_r a)}{k_r a \mathrm{J}_0(k_r a)(1 - v_{13}n) - \mathrm{J}_1(k_r a)(1 - v_{12} - 2v_{13}n)}\right]. \tag{25}$$

2.2. THE EQUIVALENT LONGITUDINAL VIBRATION OF THE PIEZOELECTRIC CERAMIC DISK RESONATOR IN COUPLED VIBRATION

From equations (3) and (4), one has

$$S_z = (s_{33}^E + s_{13}^E/n)T_z + d_{33}E_3, \qquad E_3 = [D_3 - (d_{33} + d_{31}/n)T_z]/\epsilon_{33}^T. \tag{26, 27}$$

Substituting equation (27) into (26) yields

$$T_z = E_z(S_z - d_{33}D_3/\epsilon_{33}^T). \tag{28}$$

Here

$$E_z = [s_{33}^E + s_{13}^E/n - (d_{33}/\epsilon_{33}^T)(d_{33} + d_{31}/n)]^{-1}$$

is called the equivalent longitudinal elastic constant of the disk resonator in coupled vibration. Let $v_{31} = -s_{13}^E/s_{33}^E$, E_z can be rewritten as

$$E_z = 1/s_{33}^E[1 - v_{31}/n - K_{33}^2(1 - \lambda_{31}/n)]. \tag{29}$$

Substituting equation (28) into equation (6) yields

$$\rho\, \partial^2 U_z/\partial t^2 = E_z \partial^2 U_z/\partial z^2. \tag{30}$$

Substituting $U_z = U_{za}\exp(j\omega t)$ into equation (30) yields

$$d^2U_{za}/dz^2 + k_z^2U_{za} = 0. \quad (31)$$

Here $k_z = \omega/V_z$, $V_z = (E_z/\rho)^{1/2}$, k_z and V_z are the equivalent longitudinal wave number and sound speed of the disk resonator in coupled vibration. The solution to equation (31) is

$$U_{za} = A_z\sin(k_zz) + B_z\cos(k_zz). \quad (32)$$

When the two end surfaces of the disk resonator are free from external forces, the constants A_z and B_z can be obtained. Substituting the expressions of A_z and B_z into equation (32) yields

$$U_{za} = (d_{33}D_3/\epsilon_{33}^Tk_z)\,[\sin(k_zz) + (\cos(k_zl) - 1)/\sin(k_zl)\cos(k_zz)]. \quad (33)$$

Substituting equation (33) into (28) and then (28) into (27) yields

$$E_3 = \frac{1}{\epsilon_{33}^T}\left\{D_3 - \left(d_{33} + \frac{d_{31}}{n}\right)E_z\frac{d_{33}D_3}{\epsilon_{33}^T}\left[\cos(k_zz) - \frac{\cos(k_zl) - 1}{\sin(k_zl)}\sin(k_zz) - 1\right]\right\}. \quad (34)$$

If I_{33} and V_{33} are the current and voltage of the resonator in coupled vibration, one has

$$I_{33} = j\omega D_3\pi a^2, \qquad V_{33} = \int_0^l E_3\,dz. \quad (35, 36)$$

Using equations (34) and (36), the equivalent longitudinal admittance of the resonator in coupled vibration Y_{33} can be obtained:

$$Y_{33} = I_{33}/V_{33} = (j\omega\pi a^2\epsilon_{33}^T/l)(1 - (K_{33}')^2)/[1 - (K_{33}')^2tg(k_zl/2)/(k_zl/2)] \quad (37)$$

Here

$$(K_{33}')^2 = d_{33}(d_{33} + d_{31}/n)/\epsilon_{33}^T(s_{33}^E + s_{13}^E/n)$$

and K_{33}' is the equivalent longitudinal electro–mechanical coupling coefficient of the resonator in coupled vibration. It can be rewritten as

$$(K_{33}')^2 = K_{33}^2(1 - \lambda_{31}/n)/(1 - v_{31}/n). \quad (38)$$

When $n = \infty$, equations (29) and (38) can be reduced to the forms:

$$E_z = 1/s_{33}^E(1 - K_{33}^2) = 1/s_{33}^D, \qquad K_{33}' = K_{33}. \quad (39, 40)$$

It can be seen that in this case, the coupled vibration of the piezoelectric ceramic disk resonator is reduced to the one-dimensional longitudinal vibration of the slender piezoelectric ceramic rod.

2.3. THE RESONANCE FREQUENCY EQUATIONS FOR THE PIEZOELECTRIC CERAMIC DISK RESONATOR IN COUPLED VIBRATION

When the resonator is excited by an external electric field, it will vibrate. If the voltage applied to the resonator is V_3, the current into it is I_3, the admittance Y_3 of the resonator in coupled vibration can be obtained as

$$Y_3 = I_3/V_3. \tag{41}$$

For the disk resonator of finite dimension, there exist two kinds of vibrations, i.e., the equivalent radial and longitudinal vibrations. According to the above analysis, it can be seen that

$$I_3 = I_{31} + I_{33}, \qquad V_3 = V_{31} = V_{33} \qquad \text{(42(a), (b))}$$

Therefore, the admittance of the resonator in coupled vibration can be derived as

$$Y_3 = Y_{31} + Y_{33} = j\omega\pi a^2\epsilon_{33}^T/l\,[(1-(K'_{33})^2)/(1-(K'_{33})^2 tg(k_z l/2)/(k_z l/2)) + 1 - (K'_p)^2 + (K'_p)^2 \frac{(1+v_{12})J_1(k_r a)}{k_r a J_0(k_r a)(1-v_{13}n) - J_1(k_r a)(1-v_{12}-2v_{13}n)}]. \tag{43}$$

When the admittance of the resonator has a maximal value, the resonator will resonate. Therefore, the resonance frequency equations for the resonator in coupled vibration can be obtained:

$$k_r a J_0(k_r a)(1-v_{13}n) - J_1(k_r a)(1-v_{12}-2v_{13}n) = 0, \tag{44}$$

$$1-(K'_{33})^2\tan(k_z l/2)/(k_z l/2) = 0. \tag{45}$$

In equations (44) and (45), there are two unknowns, the mechanical coupling coefficient and the angular frequency. When the material parameters and the dimensions of the resonator are given, the resonance frequency of the resonator in coupled vibration can be computed from these two equations. However, since equations (44) and (45) are transcendental equations, it is impossible to find analytical solutions. Therefore, numerical methods must be used. In solving these two transcendental equations, it is found that for certain vibrational modes (for the fundamental mode, the first roots of the equations (44) and (45) are used), there exist two sets of solutions, i.e., two resonance frequencies can be found. Considering that the coupled vibration of the resonator includes two vibrational modes, it is obvious that these two frequencies are the resonance frequencies of the resonator in longitudinal and radial vibrations. One is the first longitudinal-dominating coupled mode, and the other is the first radial-dominating coupled mode. It can be seen that these two resonance frequencies are different from those calculated from the one-dimensional theory for the resonator in longitudinal or radial vibrational mode. The reason is that in this paper the interaction between the longitudinal and radial vibrations is considered. It can also be seen that when the geometrical dimensions of the resonator satisfy certain conditions (for example, when the thickness l is much larger or smaller that its radius a), the interaction can be ignored, and the results from equations (44) and (45) are the same as those from one-dimensional theory.

2.4. ANALYSIS OF SOME LIMITING VIBRATIONAL MODES OF THE DISK RESONATOR

For the disk resonator of finite dimension, its vibration is a coupled one. However, in practical cases, when the dimensions of the resonator satisfy certain conditions, the vibration can be simplified, the one-dimensional theory can be used. It can be demonstrated in the following that some one-dimensional vibrational modes, such as the longitudinal vibration of slender piezoelectric ceramic rod, or the radial vibration of a thin disk, can be derived from the theory developed in this paper.

2.4.1 *The longitudinal vibration of a slender piezoelectric ceramic rod*

In this case, the length of the resonator is much larger than its lateral dimension, the longitudinal stress T_z is much larger than the radial and tangential stresses, and the mechanical coupling coefficient limits infinity. From the above analysis and equations, the following expressions can be obtained:

$$V_{0z} = (E_z/\rho)^{1/2} = [1/(s_{33}^D \rho)]^{1/2}, \qquad 1 - (K_{33})^2 \tan(k_{0z}l/2)/(k_{0z}l/2) = 0. \quad (46, 47)$$

Here $k_{0z} = \omega/V_{0z}$. It is obvious that this vibrational mode is the longitudinal vibration of a slender piezoelectric ceramic rod polarized in the longitudinal direction.

2.4.2 *The radial vibration of a slender piezoelectric ceramic rod*

For this vibrational mode, the longitudinal stress exists, but the axial strain does not depend on the longitudinal position. From equation (26), one has

$$E_z = \infty, \qquad n = (v_{31} - \lambda_{31} K_{33}^2)/(1 - K_{33}^2) \quad (48, 49)$$

Substituting equation (49) into equations (21), (22) and (44) yields

$$E_r = \frac{1 - v_{13}v_{31} - K_{33}^2(1 - v_{13}\lambda_{31})}{s_{11}^E(1 + v_{12})[1 - v_{12} - 2v_{13}v_{31} - K_{33}^2(1 - v_{12} - 2v_{13}\lambda_{31})]}, \quad (50)$$

$$(K_p')^2 = K_p^2 \frac{\lambda_{31} - v_{31}}{1 - v_{12} - 2v_{13}v_{31} - K_{33}^2(1 - v_{12} - 2v_{13}\lambda_{31})}, \quad (51)$$

$$k_r a \mathrm{J}_0(k_r a)[1 - v_{13}v_{31} - K_{33}^2(1 - v_{13}\lambda_{31})] - \mathrm{J}_1(k_r a)$$
$$[1 - v_{12} - 2v_{13}v_{31} - K_{33}^2 \times (1 - v_{12} - 2v_{13}\lambda_{31})] = 0. \quad (52)$$

If the resonator is an isotropic slender rod rather than a piezoelectric ceramic rod, $K_{33} = 0$, $v_{13} = v_{31} = v$, $s_{11}^E = 1/E$. From these expressions, on deduces

$$E_r = E(1 - v)/(1 + v)(1 - 2v), \qquad k_r a \mathrm{J}_0(k_r a)(1 - v) - \mathrm{J}_1(k_r a)(1 - 2v) = 0. \quad (53, 54)$$

It is obvious that equations (53) and (54) are the same as those of previous studies [13, 14]. Therefore, the limiting vibrational mode of the plane radial vibration of a slender piezoelectric ceramic rod is one of the vibration modes of the disk resonator in coupled vibration.

2.4.3 *The radial vibration of a thin piezoelectric ceramic disk*

In this case, the radial and tangential stresses exist, but the longitudinal stress disappears. Therefore, $n = 0$. From the above analysis, one has

$$V_r = (E_r/\rho)^{1/2} = [1/s_{11}^E(1 - \nu_{12}^2)\rho]^{1/2}, \qquad k_r a J_0(k_r a) - J_1(k_r a)(1 - \nu_{12}) = 0. \tag{55, 56}$$

It is obvious that this is the ideal plane radial vibration of a thin piezoelectric ceramic disk.

2.4.4 *The thickness extensional vibration of a thin piezoelectric ceramic disk*

For this vibrational mode, the radial and tangential stresses exist, but the radial and tangential strains disappear. Using equation (10), one has

$$n = (1 - \nu_{12})/2\nu_{13}. \tag{57}$$

Substituting equation (57) into the above expressions produces

$$E_z = (1 - \nu_{12})/s_{33}^E[1 - \nu_{12} - 2\nu_{13}\nu_{31} - K_{33}^2(1 - \nu_{12} - 2\nu_{13}\lambda_{31})]. \tag{58}$$

This is the thickness extensional vibration of a thin piezoelectric ceramic disk. When the resonator is an isotropic thin disk, equations (57) and (58) can be reduced to the results of a previous study [13].

From the above analysis, it can be seen that when the geometrical dimensions of the resonator satisfy certain conditions, the coupled vibration of a thin disk resonator can be reduced to some limiting vibrational mode. In these cases, one dimensional theory can be used, and the analysis can be simplified greatly.

3. EXPERIMENTAL RESULTS AND CONCLUSIONS

3.1. EXPERIMENTAL RESULTS

The resonance frequencies of some piezoelectric ceramic disk resonators were measured using the transmission line method under small excitation as shown in Figure 2. In the figure, PZT is the piezoelectric ceramic disk resonators to be measured. The condition that R_t is smaller than R_i is required for the accurate measurement of the resonance frequencies of the resonators. The frequency of the electric sine signal produced by the signal generator was changed until the output of the oscilloscope was maximised. The frequency corresponding to this maximal output of the oscilloscope is the maximal transmission frequency. Under the first approximation, this maximal transmission frequency is the resonance frequency

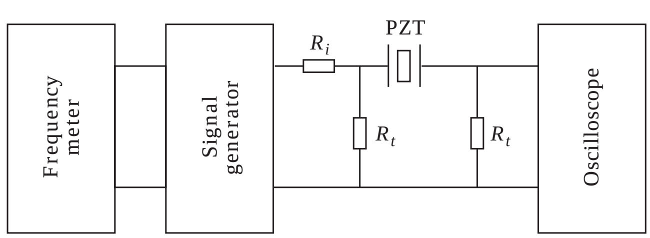

Figure 2. Circuit diagram of the experimental set up

of the resonator to be measured. In the experiment, the oscilloscope is HP Model 54601A. This method is the commonly used one to measure the resonance frequency of piezoelectric ceramic resonators. It is accurate enough to explain the experimental error with the technique and the equipment. The piezoelectric ceramic material is an equivalent of PZT-4. Its standard material parameters are used in the computation. The measured results are listed in Table 1, where f_r and f_z are the computed resonance frequencies of the disk resonator in coupled fundamental vibrational mode. For comparison, the resonance frequencies f_{1r} and f_{1z} of the disk resonator in thickness extensional and radial vibrations computed from one dimensional theory are also shown in Table 1; f_{rm} and f_{zm} are the measured results. It can be seen that the measured frequencies are in good agreement with the computed results, and the results from the theory of coupled vibration are in better agreement with the measured results than those from one-dimensional theory. From Table 1, it can be seen that the computed resonance frequency f_r for the radial-dominating coupled mode is lower than the computed resonance frequency f_{1r} for the plane radial vibration of the thin disk resonator, while the computed resonance frequency f_z for the longitudinal-dominating coupled mode is higher than the computed resonance frequency f_{1z} for the one-dimensional longitudinal vibration of a slender piezoelectric ceramic rod. This can be explained as follows. First, for the plane radial vibration of a thin disk, its circumferential boundary is free from external force. When its thickness is increased, coupled vibration is created, the energy for the longitudinal vibration is introduced, and the resonance frequency for the radial-dominating coupled vibration mode is decreased. Second, for the ideal thickness vibration of a thin disk resonator, its circumferential boundary is clamped, the radial strain is zero, while the longitudinal strain has a certain value. This implies theoretically that the lateral dimension of the resonator is very large, and the mass of the resonator also has a maximal value. When the thickness of the resonator is increased, the coupled vibration is produced, and the condition of clamped boundary is no longer suitable for the resonator. Compared with the ideal thickness vibration of a thin disk resonator whose mass has a maximal value, the mass of the resonator in coupled vibration will decrease. Therefore, the resonance frequency for the longitudinal-dominating coupled mode is increased when compared with the result of one-dimensional theory.

As for the frequency error, it is considered that the following factors should be taken into account: (1) The standard material parameters are different from the

TABLE 1

The measured resonance frequencies of the piezoelectric ceramic disk resonators

l (mm)	a (mm)	f_{1r} (kHz)	f_{1z} (kHz)	f_r (kHz)	f_z (kHz)	f_{rm} (kHz)	f_{zm} (kHz)
2	30	38·24	1005·58	38·21	1030·71	37·15	1018·43
5	30	38·24	402·25	38·08	415·27	37·01	419·22
6	30	38·24	335·19	38·04	346·26	36·95	341·62
8	30	38.24	251·39	37·90	260·51	36·63	264·33

practical values. (2) The mechanical coupling coefficient is considered as a constant. However, the mechanical coupling coefficient is different at different positions in the resonator. (3) The longitudinal and radial extensional vibrations in the resonator are supposed. However, when the disk resonator is a short cylinder or a thick disk, shearing and other strains may exist in the resonator.

3.2. CONCLUSIONS

In this paper, the coupled vibration of a piezoelectric ceramic disk resonator has been studied. An approximate analytical method is developed to analyze the complex coupled vibration. To sum up the above analysis, the following conclusions can be drawn:

(1) When the mechanical coupling coefficient is introduced, the complex coupled vibration of the disk resonator can be divided into two equivalent extensional vibrations, one being the longitudinal vibration and the other the plane radial vibration. However, these two vibrations are not independent of each other.

(2) The resonance frequency equations have been derived. The resonance frequencies of the resonator in coupled vibration can be obtained from the theory of this paper. These two frequencies correspond to the coupled longitudinal and radial vibrations in the resonator.

(3) The coupled vibration of a disk resonator is complex. Apart from the fundamental mode and the higher modes, there exist vibrations in different directions and their interaction. However, when the geometrical dimensions of the disk resonator satisfy certain conditions, the coupled vibration of disk resonator can be simplified. It has been demonstrated that some limiting one-dimensional vibration modes, such as the plane radial vibration of a thin disk and the longitudinal vibration of a slender rod can be obtained using the theory of this paper.

(4) The method presented in this paper is an approximate method. It neglects the shearing and other strains. On the other hand, the mechanical coupling coefficient is considered as a constant for a certain vibrational mode.

(5) The method presented in this paper can be used to analyze and compute the resonance frequency of the piezoelectric ceramic disk resonators in coupled longitudinal and radial vibration. The coupled mode shapes can not be calculated using this method. However, numerical methods such as the finite element method can be used to accomplish this task.

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APPENDIX: LIST OF NOTATION

l thickness of the disk resonator
a radius of the disk resonator
S_{ij}^E elastic compliant constant at constant electric field
d_{mn} piezoelectric strain constant
E_3 external electric field along longitudinal axis
D_3 electric flux density
S_i strain tensor
T_i stress tensor
ϵ_{33}^T dielectric constant at constant stress
ρ density
n mechanical coupling coefficient between different vibrational modes
E_r equivalent radial elastic constant in coupled vibration
U_i displacement amplitude
k_r equivalent radial wave number in coupled vibration
V_r equivalent radial sound speed in coupled vibration
K_p' equivalent plane electro-mechanical coupling coefficient of disk resonator in coupled vibration
K_p equivalent plane electro-mechanical coupling coefficient of thin disk resonator in ideal radial vibration
Y_{ij} electric admittance
E_z equivalent longitudinal elastic constant in coupled vibration
V_z equivalent longitudinal sound speed in coupled vibration
k_z equivalent longitudinal wave number in coupled vibration

I_{ij} electric current
V_{ij} electric voltage
K'_{33} equivalent longitudinal electro-mechanical coupling coefficient of the disk resonator in coupled vibration
K_{33} equivalent longitudinal electro-mechanical coupling coefficient of a slender piezoelectric ceramic rod in one-dimensional vibration
ω angular frequency